Government of Western Australia
Curriculum Council

WACE
Western Australian Certificate of Education

MATHEMATICS: SPECIALIST

3C/3D
Calculator-free

WACE Examination 2010

Final Marking Key

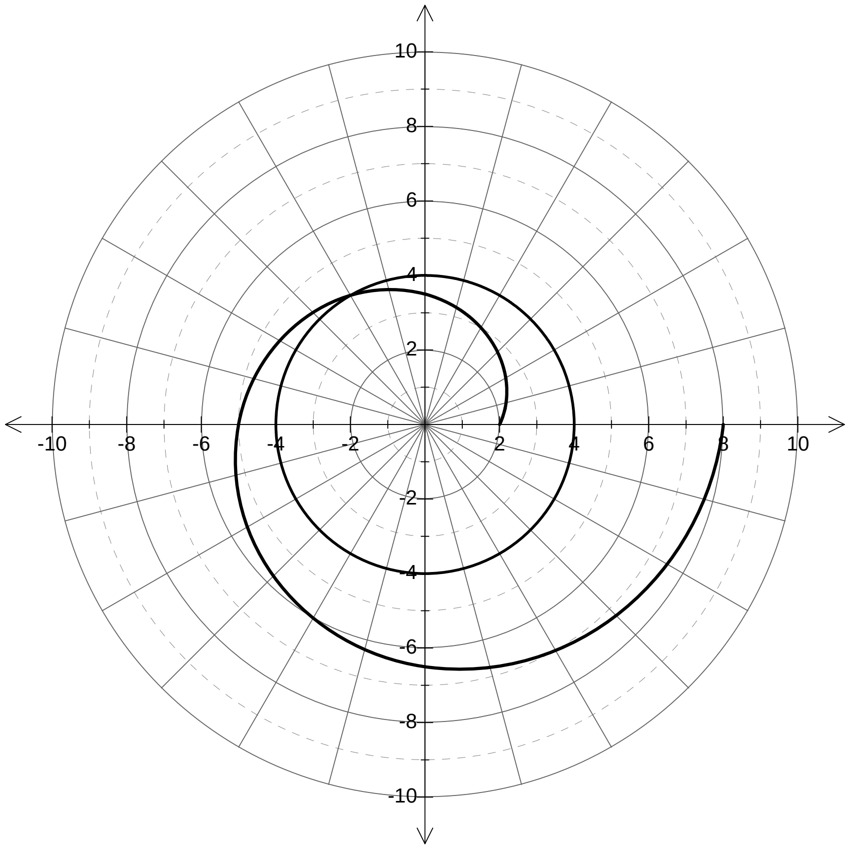
This 'stand alone' version of the WACE Examination 2010 Final Marking Key is provided on an interim basis.

The Standards Guide for this examination will include the examination questions, marking key, question statistics and annotated candidate responses. When the Standards Guide is published, this document will be removed from the website.

Section One: Calculator-free **40 Marks**

Question 1 **(4 marks)**

(a) On the axes below, draw the graphs of $r = 4$ and $r = \frac{3\theta}{\pi} + 2$, for $0 \le \theta \le 2\pi$. (3 marks)

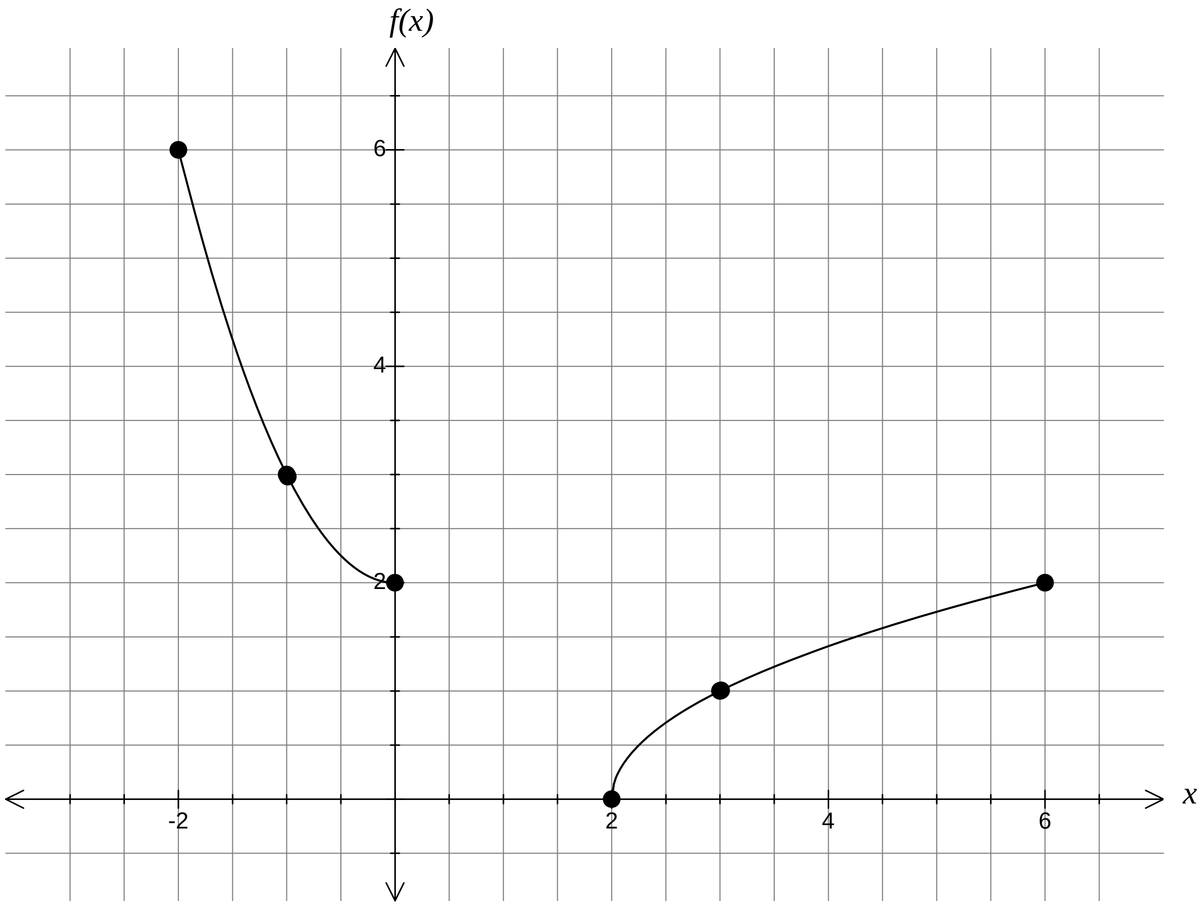
Solution

Specific behaviours
✓ draws the circle $r = 4$
✓ starts the spiral at the point $(2,\ 0)$
✓ draws the spiral accurately with all correct intercepts

(b) Write down the polar coordinates of the point where these graphs intersect. (1 mark)

Solution
$\left(4,\ \frac{2\pi}{3}\right)$
Specific behaviours
✓ correctly solves $4 = \frac{3\theta}{\pi} + 2$ for θ

Question 2 **(7 marks)**

The points $(-2,\ 6)$, $(-1,\ 3)$ and $(0,\ 2)$ on the graph of the function $f(x) = x^2 + 2$ are moved by a linear transformation as shown in the diagram below.



(a) State the appropriate transformation matrix. (1 mark)

Solution
Transformation matrix is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
Specific behaviours
✓ states the correct transformation matrix

(b) Under a second transformation, $(2,\ 0)$ and $(3,\ 1)$ become $(2,\ 0)$ and $(5,\ 1)$ respectively.

(i) Determine the matrix that will achieve this. (2 marks)

Solution
$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$ Hence $2a = 2$, $2c = 0$, $3a + b = 5$ and $3c + d = 1$ i.e. $a = 1$, $c = 0$, $b = 2$ and $d = 1$ Transformation matrix is $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ **Or:** This is a shear parallel to the x-axis so the matrix is $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ for some value of a. By inspection $a = 2$
Specific behaviours
✓ correctly solves for a and c ✓ correctly solves for b and d **Or:** ✓ recognises that the transformation is a shear parallel to the x-axis ✓ correctly solves for a

(ii) State the new coordinates of $(6,\ 2)$. (1 mark)

Solution
$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$ New coordinates: $(10,\ 2)$
Specific behaviours
✓ correctly states the new coordinates for $(6,\ 2)$, i.e. $(10,\ 2)$

(c) Find the transformation matrix which will transform the points from their final positions in part (b) back to their original positions. (3 marks)

Solution
Matrix is the inverse of $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$ i.e. required matrix is $\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$
Specific behaviours
✓ orders the two transformation matrices correctly ✓ correctly multiplies the matrices ✓ derives the correct inverse

Question 3 **(7 marks)**

Find the following indefinite integrals:

(a) $\int \frac{e^{2x}}{3+e^{2x}}\,dx$ (2 marks)

Solution
$\int \frac{e^{2x}}{3+e^{2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{3+e^{2x}}\,dx$ $= \frac{1}{2}\ln\left(3+e^{2x}\right)+c$
Specific behaviours
✓ recognises the format as: $a\int \frac{f'(x)}{f(x)}\,dx$ ✓ integrates correctly

(b) $\int \left(x^e - \frac{1}{\cos^2 3x}\right)dx$ (2 marks)

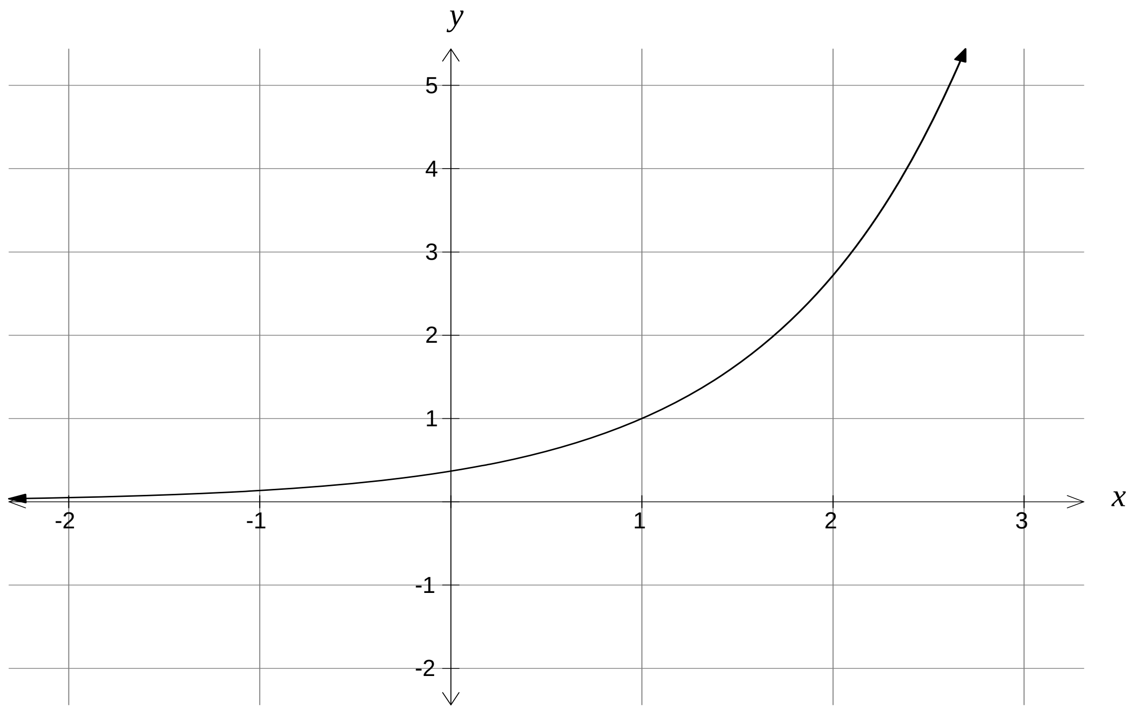
Solution
$\int\left(x^e - \frac{1}{\cos^2 3x}\right)dx = \frac{x^{e+1}}{e+1} + \frac{\tan 3x}{3} + c$
Specific behaviours
✓ recognises $\frac{1}{\cos^2 3x}$ as $\sec^2 3x$, the derivative of $\frac{\tan 3x}{3}$ ✓ integrates the other terms correctly

(c) $\int \cos x\left(1-\cos x\right)dx$ (3 marks)

Solution
$\int \cos x\left(1-\cos x\right)dx = \int\left(\cos x - \cos^2 x\right)dx$ $= \int \cos x\ dx - \frac{1}{2}\int\left(1+\cos 2x\right)dx$ $= \sin x - \frac{x}{2} - \frac{\sin 2x}{4} + c$
Specific behaviours
✓ transforms the integral using the trigonometric identity $\cos^2 x = \frac{1}{2}\left(1+\cos 2x\right)$ ✓✓integrates correctly

Question 4 **(4 marks)**

The graph of $y = e^{(x-1)}$ is shown below.

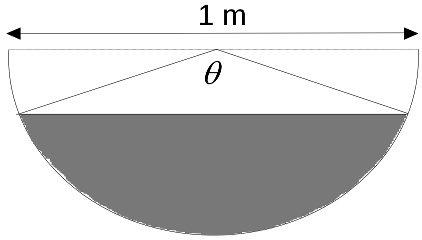


Calculate the exact area between the graphs of $y = e^{(x-1)}$, $y = 2 - x$ and the two axes.

Solution
$A = \int_0^1 e^{(x-1)}dx + \int_1^2 (2-x)dx$ i.e. $A = \left[e^{(x-1)}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$ i.e. $A = 1 - e^{-1} + (4-2) - \left(2 - \frac{1}{2}\right)$ i.e. $A = \frac{3}{2} - \frac{1}{e}$
Specific behaviours
✓ recognises that the area is the sum of two integrals   ✓ identifies the correct limits for each integral   ✓ correctly integrates   ✓ substitutes and simplifies

Question 5 **(9 marks)**

A farmer has a water trough of length 8 metres with a semi-circular cross-section of diameter 1 m, as shown below.



(a) Show that the volume (V m^3) of water in the trough is given by
$V = \theta - \sin\theta$, where θ is measured in radians. (1 mark)

Solution
$A = \frac{1}{2}r^2(\theta - \sin\theta)$ i.e. $A = \frac{1}{2}\left(\frac{1}{2}\right)^2(\theta - \sin\theta) = \frac{1}{8}(\theta - \sin\theta)$ Thus $V = 8 \times \frac{1}{8}(\theta - \sin\theta) = (\theta - \sin\theta)$
Specific behaviours
✓ correctly substitutes for the length of the radius and the length of the trough

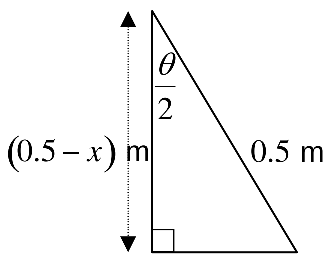
Water is pumped into the trough at a constant rate of 0.1 m^3 per minute.

(b) Use the above to determine an expression for $\frac{d\theta}{dt}$. (2 marks)

Solution
$\frac{dV}{dt} = (1-\cos\theta)\frac{d\theta}{dt}$ But $\frac{dV}{dt} = 0.1$ Thus $\frac{1}{10} = (1-\cos\theta)\frac{d\theta}{dt}$ i.e. $\frac{d\theta}{dt} = \frac{1}{10(1-\cos\theta)}$
Specific behaviours
✓ calculates $\frac{dV}{d\theta}$ ✓ recognises that $\frac{dV}{dt} = 0.1$ and uses the chain rule to deduce $\frac{d\theta}{dt}$

Let x m be the depth of the water in the trough.

(c) Use the triangle pictured here to express x in terms of θ. (2 marks)



Solution
$\cos\frac{\theta}{2} = \frac{0.5-x}{0.5} = 1-2x$ i.e. $x = \frac{1}{2}\left(1-\cos\frac{\theta}{2}\right)$
Specific behaviours
✓ correctly expresses $\cos\frac{\theta}{2}$ in terms of x ✓ rearranges to express x in terms of θ

(d) Show that $\frac{dx}{dt}=\frac{1}{4}\sin\left(\frac{\theta}{2}\right)\frac{d\theta}{dt}$. (1 mark)

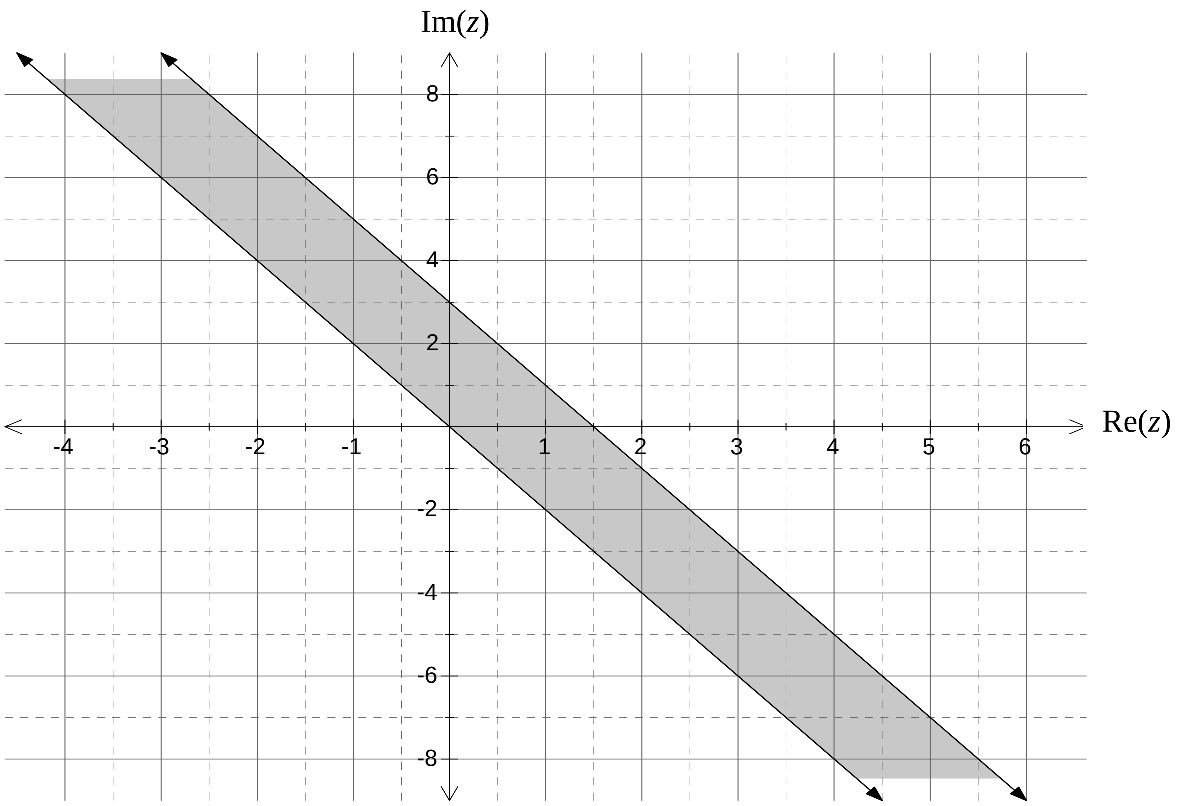
Solution
$x=\frac{1}{2}\left(1-\cos\frac{\theta}{2}\right)=\frac{1}{2}-\frac{1}{2}\cos\frac{\theta}{2}$ Thus $\frac{dx}{dt}=\frac{dx}{d\theta}\times\frac{d\theta}{dt}=\frac{1}{4}\sin\left(\frac{\theta}{2}\right)\frac{d\theta}{dt}$
Specific behaviours
✓ correctly differentiates x with respect to θ

(e) Determine, exactly, the rate at which the water level is rising at the instant when the water is 25 cm deep. (3 marks)

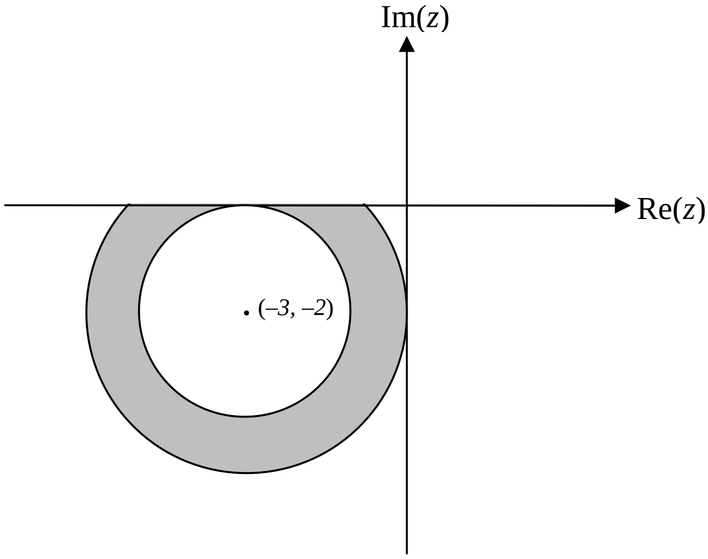
Solution
$\cos\frac{\theta}{2}=1-2x=1-0.5=0.5$ i.e. $\frac{\theta}{2}=\frac{\pi}{3}$ i.e. $\theta=\frac{2\pi}{3}$ $\frac{dx}{dt}=\frac{1}{4}\sin\left(\frac{\theta}{2}\right)\frac{d\theta}{dt}=\frac{1}{4}\times\frac{\sqrt{3}}{2}\times\frac{1}{10(1+0.5)}$ i.e. $\frac{dx}{dt}=\frac{\sqrt{3}}{120}$ m/min
Specific behaviours
✓ finds the correct value for $\cos\frac{\theta}{2}$ when $x=25$ cm ✓ correctly determines the value of θ ✓ substitutes in $\frac{dx}{dt}=\frac{1}{4}\sin\left(\frac{\theta}{2}\right)\frac{d\theta}{dt}$ and simplifies

Question 6 **(5 marks)**

(a) Sketch, on the complex plane below, the region defined by $0 \le 2\,\mathrm{Re}(z) + \mathrm{Im}(z) \le 3$.
(2 marks)

Solution

Specific behaviours
✓ correctly draws the 2 lines ✓ shades between the lines

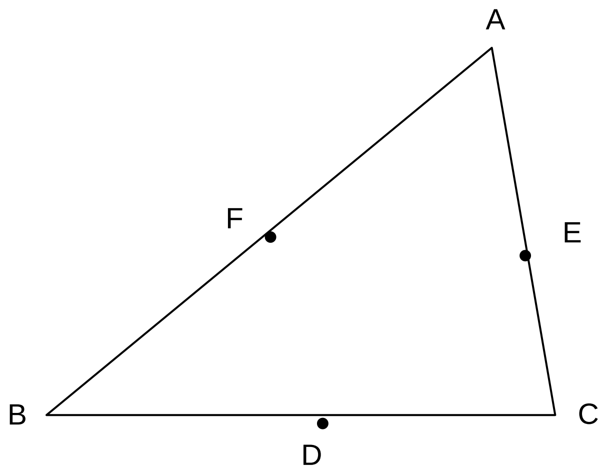
(b) State the inequalities that together describe the set of points illustrated by the shading below. (3 marks)



Solution
$2 \le \lvert z+3+2i \rvert \le 3$ and $\text{Im}(z) \le 0$
Specific behaviours
✓ recognises the format for complex circles and states $\lvert z+3+2i \rvert$ correctly ✓ correctly states upper and lower bounds ✓ recognises that $\text{Im}(z) \le 0$

Question 7 **(4 marks)**

In the diagram below, D, E and F are midpoints of the sides of the triangle ABC. Prove that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \underset{\sim}{0}$.



Solution
$\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\left(\overrightarrow{AB}+\frac{1}{2}\overrightarrow{BC}\right)+\left(\overrightarrow{BC}+\frac{1}{2}\overrightarrow{CA}\right)+\left(\overrightarrow{CA}+\frac{1}{2}\overrightarrow{AB}\right)$ i.e. $\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\left(\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}\right)+\frac{1}{2}\left(\overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB}\right)$ i.e. $\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\underset{\sim}{0}+\frac{1}{2}\underset{\sim}{0}$ Hence result
Specific behaviours
✓✓expresses $\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}$ in terms of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ ✓ correctly rearranges   ✓ deduces the required result   **Or:**   ✓ defines two base vectors, e.g. $\underset{\sim}{a}$ and $\underset{\sim}{b}$ ✓✓expresses $\overrightarrow{AD}$, $\overrightarrow{BC}$ and $\overrightarrow{CF}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ ✓ deduces the required result

End of questions